Higher

Mathematics

UNIT 3 OUTCOME 2

Further Calculus

Contents

HSN23200

OUTCOME 2

Further Calculus

1 Differentiating sin*x* and cos*x*

In order to differentiate expressions involving trigonometric functions, we use the following rules:

$$\frac{d}{dx}(\sin x) = \cos x, \qquad \frac{d}{dx}(\cos x) = -\sin x.$$

These rules only work when x is an angle measured in radians. A form of these rules is given in the exam.

EXAMPLES

1. Differentiate $y = 3\sin x$ with respect to x.

$$\frac{dy}{dx} = 3\cos x.$$

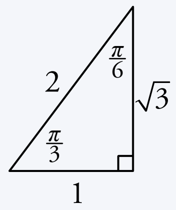
2. A function f is defined by $f(x) = \sin x - 2\cos x$ for $x \in \mathbb{R}$.

   Find $f'\left(\frac{\pi}{3}\right)$.

$$\begin{aligned} f'(x) &= \cos x - (-2\sin x) \\ &= \cos x + 2\sin x \\ f'\left(\tfrac{\pi}{3}\right) &= \cos\tfrac{\pi}{3} + 2\sin\tfrac{\pi}{3} \\ &= \tfrac{1}{2} + 2 \times \tfrac{\sqrt{3}}{2} \\ &= \tfrac{1}{2} + \sqrt{3}. \end{aligned}$$

Remember

The exact value triangle:



3. Find the equation of the tangent to the curve $y = \sin x$ when $x = \frac{\pi}{6}$.

When $x = \frac{\pi}{6}$, $y = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. So the point is $\left(\frac{\pi}{6}, \frac{1}{2}\right)$.

We also need the gradient at the point where $x = \frac{\pi}{6}$:

$$\frac{dy}{dx} = \cos x.$$

When $x = \frac{\pi}{6}$, $m_{\text{tangent}} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

Now we have the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ and the gradient $m_{\text{tangent}} = \frac{\sqrt{3}}{2}$, so:

$$\begin{aligned} y - b &= m(x-a) \\ y - \tfrac{1}{2} &= \tfrac{\sqrt{3}}{2}\left(x - \tfrac{\pi}{6}\right) \\ 2y - 1 &= \sqrt{3}x - \tfrac{\sqrt{3}\pi}{6} \\ \sqrt{3}x - 2y - \tfrac{\sqrt{3}\pi}{6} + 1 &= 0. \end{aligned}$$

2 Integrating sinx and cosx

We know the derivatives of $\sin x$ and $\cos x$, so it follows that the integrals are:

$$\int \cos x \, dx = \sin x + c, \qquad \int \sin x \, dx = -\cos x + c.$$

Again, these results only hold if x is measured in radians.

EXAMPLES

1. Find $\int (5\sin x + 2\cos x)\, dx$.

$$\int (5\sin x + 2\cos x)\, dx = -5\cos x + 2\sin x + c.$$

2. Find $\int_0^{\frac{\pi}{4}} (4\cos x + 2\sin x)\, dx$.

$$\begin{aligned} \int_0^{\frac{\pi}{4}} (4\cos x + 2\sin x)\, dx &= \left[4\sin x - 2\cos x\right]_0^{\frac{\pi}{4}} \\ &= \left(4\sin\left(\tfrac{\pi}{4}\right) - 2\cos\left(\tfrac{\pi}{4}\right)\right) - \left(4\sin 0 - 2\cos 0\right) \\ &= \left(\left(4\times\tfrac{1}{\sqrt{2}}\right) - \left(2\times\tfrac{1}{\sqrt{2}}\right)\right) - (-2) \\ &= \tfrac{4}{\sqrt{2}} - \tfrac{2}{\sqrt{2}} + 2 \\ &= \left(\tfrac{2}{\sqrt{2}} \times \tfrac{\sqrt{2}}{\sqrt{2}}\right) + 2 \\ &= \sqrt{2} + 2. \end{aligned}$$

Note
It is good practice to rationalise the denominator.

3. Find the value of $\int_0^4 \frac{1}{2}\sin x \, dx$.

$$\begin{aligned} \int_0^4 \tfrac{1}{2}\sin x\, dx &= \left[-\tfrac{1}{2}\cos x\right]_0^4 \\ &= -\tfrac{1}{2}\cos(4) + \tfrac{1}{2}\cos(0) \\ &= \tfrac{1}{2}(0{\cdot}654 + 1) \\ &= 0{\cdot}827 \quad \text{(to 3 d.p.).} \end{aligned}$$

Remember
We must use radians when integrating or differentiating trig. functions.

3 The Chain Rule

We will now look at how to differentiate composite functions, such as $f(g(x))$. If the functions f and g are defined on suitable domains, then

$$\tfrac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \times g'(x).$$

Stated simply: differentiate the outer function, the bracket stays the same, then multiply by the derivative of the bracket.

This is called the **chain rule**. You will need to remember it for the exam.

EXAMPLE

If $y = \cos\left(5x + \tfrac{\pi}{6}\right)$, find $\dfrac{dy}{dx}$.

$$\begin{aligned} y &= \cos\left(5x + \tfrac{\pi}{6}\right) \\ \frac{dy}{dx} &= -\sin\left(5x + \tfrac{\pi}{6}\right) \times 5 \\ &= -5\sin\left(5x + \tfrac{\pi}{6}\right). \end{aligned}$$

Note
The "$\times 5$" comes from $\tfrac{d}{dx}\left(5x + \tfrac{\pi}{6}\right)$.

4 Special Cases of the Chain Rule

We will now look at how the chain rule can be applied to particular types of expression.

Powers of a Function

For expressions of the form $[f(x)]^n$, where n is a constant, we can use a simpler version of the chain rule:

$$\tfrac{d}{dx}\left[(f(x))^n\right] = n[f(x)]^{n-1} \times f'(x).$$

Stated simply: the power (n) multiplies to the front, the bracket stays the same, the power lowers by one (giving $n-1$) and everything is multiplied by the derivative of the bracket $(f'(x))$.

EXAMPLES

1. A function f is defined on a suitable domain by $f(x)=\sqrt{2x^2+3x}$.
   Find $f'(x)$.

$$f(x)=\sqrt{2x^2+3x}=\left(2x^2+3x\right)^{\frac{1}{2}}$$

$$\begin{aligned} f'(x) &= \tfrac{1}{2}\left(2x^2+3x\right)^{-\frac{1}{2}}\times(4x+3) \\ &= \tfrac{1}{2}(4x+3)\left(2x^2+3x\right)^{-\frac{1}{2}} \\ &= \frac{4x+3}{2\sqrt{2x^2+3x}}. \end{aligned}$$

2. Differentiate $y=2\sin^4 x$ with respect to x.

$$y=2\sin^4 x=2(\sin x)^4$$

$$\begin{aligned} \frac{dy}{dx} &= 2\times 4(\sin x)^3\times\cos x \\ &= 8\sin^3 x\cos x. \end{aligned}$$

Powers of a Linear Function

The rule for differentiating an expression of the form $(ax+b)^n$, where a, b and n are constants, is as follows:

$$\tfrac{d}{dx}\left[(ax+b)^n\right]=an(ax+b)^{n-1}.$$

EXAMPLES

3. Differentiate $y=(5x+2)^3$ with respect to x.

$$y=(5x+2)^3$$

$$\begin{aligned} \frac{dy}{dx} &= 3(5x+2)^2\times 5 \\ &= 15(5x+2)^2. \end{aligned}$$

4. If $y = \dfrac{1}{(2x+6)^3}$, find $\dfrac{dy}{dx}$.

$$y = \frac{1}{(2x+6)^3} = (2x+6)^{-3}$$

$$\begin{aligned}\frac{dy}{dx} &= -3(2x+6)^{-4} \times 2 \\ &= -6(2x+6)^{-4} \\ &= -\frac{6}{(2x+6)^4}.\end{aligned}$$

5. A function f is defined by $f(x) = \sqrt[3]{(3x-2)^4}$ for $x \in \mathbb{R}$. Find $f'(x)$.

$$f(x) = \sqrt[3]{(3x-2)^4} = (3x-2)^{\frac{4}{3}}$$

$$\begin{aligned}f'(x) &= \tfrac{4}{3}(3x-2)^{\frac{1}{3}} \times 4 \\ &= \tfrac{16}{3}\sqrt[3]{(3x-2)}.\end{aligned}$$

Trigonometric Functions

The following rules can be used to differentiate trigonometric functions.

$$\tfrac{d}{dx}\big[\sin(ax+b)\big] = a\cos(ax+b) \qquad \tfrac{d}{dx}\big[\cos(ax+b)\big] = -a\sin(ax+b)$$

These are given in the exam.

EXAMPLE

6. Differentiate $y = \sin(9x+\pi)$ with respect to *x*.

$$\frac{dy}{dx} = 9\cos(9x+\pi).$$

5 A Special Integral

The method for integrating an expression of the form $(ax+b)^n$ is:

$$\int (ax+b)^n \ dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \qquad \text{where } a \neq 0 \text{ and } n \neq -1.$$

Stated simply: raise the power (n) by one, divide by the new power and also divide by the derivative of the bracket ($a(n+1)$), add c.

EXAMPLES

1. Find $\int (x+4)^7 \ dx$.

$$\begin{aligned}\int (x+4)^7 \ dx &= \frac{(x+4)^8}{8\times 1} + c \\ &= \frac{(x+4)^8}{8} + c.\end{aligned}$$

2. Find $\int (2x+3)^2 \ dx$.

$$\begin{aligned}\int (2x+3)^2 \ dx &= \frac{(2x+3)^3}{3\times 2} + c \\ &= \frac{(2x+3)^3}{6} + c.\end{aligned}$$

3. Find $\int \frac{1}{\sqrt[3]{5x+9}} \ dx$ where $x \neq -\frac{9}{5}$.

$$\begin{aligned}\int \frac{1}{\sqrt[3]{5x+9}} \ dx &= \int \frac{1}{(5x+9)^{\frac{1}{3}}} \ dx \\ &= \int (5x+9)^{-\frac{1}{3}} \ dx \\ &= \frac{(5x+9)^{\frac{2}{3}}}{\frac{2}{3}\times 5} + c \\ &= \frac{\sqrt[3]{5x+9}^{\,2}}{\frac{10}{3}} + c \\ &= \frac{3}{10}\sqrt[3]{5x+9}^{\,2} + c.\end{aligned}$$

4. Evaluate $\int_0^3 \sqrt{3x+4}\, dx$ where $x \geq -\frac{4}{3}$.

$$\begin{aligned}\int_0^3 \sqrt{3x+4}\, dx &= \int_0^3 (3x+4)^{\frac{1}{2}}\, dx \\ &= \left[\frac{(3x+4)^{\frac{3}{2}}}{\frac{3}{2}\times 3}\right]_0^3 \\ &= \left[\tfrac{2}{9}\sqrt{(3x+4)^3}\right]_0^3 \\ &= \tfrac{2}{9}\sqrt{(3\times 3+4)^3} - \tfrac{2}{9}\sqrt{(3\times 0+4)^3} \\ &= \tfrac{2}{9}\sqrt{13^3} - \tfrac{2}{9}\sqrt{4^3} \\ &= \tfrac{2}{9}\left(\sqrt{13}^3 - 8\right) \qquad \text{(or } 8{\cdot}638 \text{ to 3 d.p.).}\end{aligned}$$

Note
Changing powers back into roots here makes it easier to evaluate the two brackets.

Remember
To evaluate $\sqrt{4^3}$, it is easier to work out $\sqrt{4}$ first.

Warning

Make sure you don't confuse differentiation and integration – this could lose you a lot of marks in the exam.

Remember the following rules for differentiating and integrating expressions of the form $(ax+b)^n$:

$$\frac{d}{dx}\left[(ax+b)^n\right] = an(ax+b)^{n-1},$$

$$\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c.$$

These rules will *not* be given in the exam.

Using Differentiation to Integrate

Recall that integration is just the process of undoing differentiation. So if we differentiate $f(x)$ to get $g(x)$ then we know that $\int g(x)\,dx = f(x)+c$.

EXAMPLES

5. (a) Differentiate $y=\dfrac{5}{(3x-1)^4}$ with respect to x.

(b) Hence, or otherwise, find $\displaystyle\int \frac{1}{(3x-1)^5}\,dx$.

(a) $$y=\frac{5}{(3x-1)^4}=5(3x-1)^{-4}$$

$$\frac{dy}{dx}=5\times3\times(-4)(3x-1)^{-5}$$

$$=-\frac{60}{(3x-1)^5}.$$

(b) From part (a) we know $\displaystyle\int -\frac{60}{(3x-1)^5}\,dx=\frac{5}{(3x-1)^4}+c$. So:

$$-60\int\frac{1}{(3x-1)^5}\,dx=\frac{5}{(3x-1)^4}+c$$

$$\int\frac{1}{(3x-1)^5}\,dx=-\frac{1}{60}\left(\frac{5}{(3x-1)^4}+c\right)$$

$$=-\frac{1}{12(3x-1)^4}+c_1$$ where c_1 is some constant.

Note
We could also have used the special integral to obtain this answer.

6. (a) Differentiate $y=\dfrac{1}{\left(x^3-1\right)^5}$ with respect to x.

(b) Hence, find $\displaystyle\int\frac{x^2}{\left(x^3-1\right)^6}\,dx$.

(a) $$y=\frac{1}{\left(x^3-1\right)^5}=\left(x^3-1\right)^{-5}$$

$$\frac{dy}{dx}=-5\left(x^3-1\right)^{-6}\times3x^2$$

$$=-\frac{15x^2}{\left(x^3-1\right)^6}.$$

(b) From part (a) we know $\int -\frac{15x^2}{(x^3-1)^6}\,dx = \frac{1}{(x^3-1)^5}+c$. So:

$$-15\int \frac{x^2}{(x^3-1)^6}\,dx = \frac{1}{(x^3-1)^5}+c$$

$$\int \frac{x^2}{(x^3-1)^6}\,dx = -\tfrac{1}{15}\left(\frac{1}{(x^3-1)^5}+c\right)$$

$$= -\frac{1}{15(x^3-1)^5}+c_1 \qquad \text{where } c_1 \text{ is some constant.}$$

Note
In this case, the special integral cannot be used.

6 Integrating sin(*ax* + *b*) and cos(*ax* + *b*)

Since we know the derivatives of $\sin(ax+b)$ and $\cos(ax+b)$, it follows that their integrals are:

$$\int \cos(ax+b)\;dx = \tfrac{1}{a}\sin(ax+b)+c,$$

$$\int \sin(ax+b)\;dx = -\tfrac{1}{a}\cos(ax+b)+c.$$

These are given in the exam.

EXAMPLES

1. Find $\int \sin(4x+1)\;dx$.

$$\int \sin(4x+1)\;dx = -\tfrac{1}{4}\cos(4x+1)+c.$$

2. Find $\int \cos\left(\tfrac{3}{2}x+\tfrac{\pi}{5}\right)\;dx$.

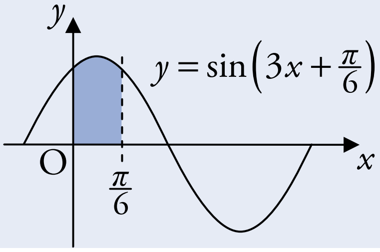
$$\int \cos\left(\tfrac{3}{2}x+\tfrac{\pi}{5}\right)\;dx = \tfrac{2}{3}\sin\left(\tfrac{3}{2}x+\tfrac{\pi}{5}\right)+c.$$

3. Find the value of $\int_0^1 \cos(2x-5)\,dx$.

$$\begin{aligned}\int_0^1 \cos(2x-5)\,dx &= \left[\tfrac{1}{2}\sin(2x-5)\right]_0^1\\ &= \tfrac{1}{2}\sin(-3)-\tfrac{1}{2}\sin(-5)\\ &= \tfrac{1}{2}(-0{\cdot}141-0{\cdot}959)\\ &= -0{\cdot}55 \;\;\text{(to 2 d.p.).}\end{aligned}$$

Remember
We must use radians when integrating or differentiating trig. functions.

4. Find the area enclosed by the graph of $y=\sin\left(3x+\frac{\pi}{6}\right)$, the x-axis and the lines $x=0$ and $x=\frac{\pi}{6}$.



$$\begin{aligned}\int_0^{\frac{\pi}{6}} \sin\left(3x+\tfrac{\pi}{6}\right)dx &= \left[-\tfrac{1}{3}\cos\left(3x+\tfrac{\pi}{6}\right)\right]_0^{\frac{\pi}{6}}\\ &= \left(-\tfrac{1}{3}\cos\left(3\left(\tfrac{\pi}{6}\right)+\tfrac{\pi}{6}\right)\right)-\left(-\tfrac{1}{3}\cos\left(3(0)+\tfrac{\pi}{6}\right)\right)\\ &= \left(\left(-\tfrac{1}{3}\right)\times\left(-\tfrac{1}{2}\right)\right)+\left(\tfrac{1}{3}\times\tfrac{\sqrt{3}}{2}\right)\\ &= \tfrac{1}{6}+\tfrac{\sqrt{3}}{6}\\ &= \frac{1+\sqrt{3}}{6}.\end{aligned}$$

So the area is $\frac{1+\sqrt{3}}{6}$ square units.

5. Find $\int 2\cos\left(\frac{1}{2}x-3\right)\,dx$.

$$\begin{aligned}\int 2\cos\left(\tfrac{1}{2}x-3\right)dx &= \frac{2}{\frac{1}{2}}\sin\left(\tfrac{1}{2}x-3\right)+c\\ &= 4\sin\left(\tfrac{1}{2}x-3\right)+c.\end{aligned}$$

6. Find $\int\left(5\cos(2x)+\sin\left(x-\sqrt{3}\right)\right)dx$.

$$\int\left(5\cos(2x)+\sin\left(x-\sqrt{3}\right)\right)dx=\tfrac{5}{2}\sin(2x)-\cos\left(x-\sqrt{3}\right)+c.$$

7. (a) Differentiate $\dfrac{1}{\cos x}$ with respect to x.

(b) Hence find $\displaystyle\int \frac{\tan x}{\cos x}\, dx$.

(a) $\dfrac{1}{\cos x} = (\cos x)^{-1}$, and $\frac{d}{dx}(\cos x)^{-1} = -1(\cos x)^{-2} \times -\sin x$

$$= \frac{\sin x}{\cos^2 x}.$$

(b) $\dfrac{\tan x}{\cos x} = \dfrac{\frac{\sin x}{\cos x}}{\cos x} = \dfrac{\sin x}{\cos^2 x}$.

From part (a) we know $\displaystyle\int \frac{\sin x}{\cos^2 x}\, dx = \frac{1}{\cos x} + c$.

Therefore $\displaystyle\int \frac{\tan x}{\cos x}\, dx = \frac{1}{\cos x} + c$.